On Bernoulli convolutions generated by second Ostrogradsky series and their fine fractal properties

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Abstract

We study properties of Bernoulli convolutions generated by the second Ostrogradsky series, i.e., probability distributions of random variables

$$\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\xi_k}{q_k}, \tag{1}$$

where q_k is a sequence of positive integers with $q_{k+1} \geq q_k(q_k+1)$, and $\{\xi_k\}$ are independent random variables taking the values 0 and 1 with probabilities p_{0k} and p_{1k} respectively. We prove that ξ has an anomalously fractal Cantor type singular distribution ($\dim_H(S_\xi) = 0$) whose Fourier-Stieltjes transform does not tend to zero at infinity. We also develop different approaches how to estimate a level of "irregularity" of probability distributions whose spectra are of zero Hausdorff dimension. Using generalizations of the Hausdorff measures and dimensions, fine fractal properties of the probability measure μ_ξ are studied in details. Conditions for the Hausdorff–Billingsley dimension preservation on the spectrum by its probability distribution function are also obtained.

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1 Introduction

About 1861 M. V. Ostrogradsky considered two algorithms for the expansions of positive real numbers in alternating series:

$$\sum_k \frac{(-1)^{k+1}}{q_1 q_2 \dots q_k}, \quad \text{where } q_k \in \mathbb{N},\ q_{k+1} > q_k, \tag{2}$$

$$\sum_k \frac{(-1)^{k+1}}{q_k}, \quad \text{where } q_k \in \mathbb{N},\ q_{k+1} \geq q_k(q_k+1) \tag{3}$$

(the first and the second Ostrogradsky series respectively). They were found by E. Ya. Remez among manuscripts and unpublished papers by M. V. Ostrogradsky [21]. These series give good rational approximations for real numbers.

One can prove (see, e.g., [19]) that for any real number $x \in [0,1]$ there exists a sequence $\{q_k = q_k(x)\}$ such that $q_{k+1} \geq q_k(q_k+1)$ and

$$x = \frac{1}{q_1(x)} - \frac{1}{q_2(x)} + \frac{1}{q_3(x)} + \dots + \frac{1}{q_k(x)} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k(x)}. \tag{4}$$

If x is irrational, then the expansion (8) is unique and it has an infinite number of terms. So, (8) establishes one-to-one mapping between the set of all infinite increasing sequences of positive integers $(q_1, q_2 \dots, q_k, \dots)$ with $q_{k+1} \geq q_k(q_k+1)$ and the set of all irrational numbers from the unit interval. If x is rational, then the expression (8) has a finite number of terms and there exist exactly two different expansions of x in form (8).

In the present paper we study properties of infinite Bernoulli convolutions generated by series of the form (3). Let us shortly recall that an infinite(general non-symmetric) Bernoulli convolutions with bounded spectra is the distribution of random series $\sum\limits_{k=1}^{\infty} \xi_k b_k$, where $\sum\limits_{k=1}^{\infty} |b_k| < \infty$, and ξ_k are independent random variables taking values 0 and 1 with probabilities p_{0k} and p_{1k} respectively. Measures of this form have been studied since 1930's from the pure probabilistic point of view as well as for their applications in harmonic analysis, in the theory of dynamical systems and in fractal analysis (see, e.g., [18] for details and references).

The main purpose of the paper is to study properties of Bernoulli convolutions generated by the second Ostrogradsky series, i.e., the probability distributions of random variables

$$\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\xi_k}{q_k}, \tag{5}$$

where q_k is an arbitrary second Ostrogradsky sequence, i.e., sequence of positive integers with $q_{k+1} \geq q_k(q_k+1)$, and $\{\xi_k\}$ is a sequence of independent random variables taking the values 0 and 1 with probabilities p_{0k} and p_{1k} respectively ($p_{0k} + p_{1k} = 1$). Since there exists a natural one-to-one correspondence between the set of irrational numbers of the unit interval and the set of infinite second Ostrogradsky sequences, we have a natural parametrization of symmetric ($p_{0k} = \frac{1}{2}$) random variables of the form (5) via the set of irrational numbers.

We prove that the distribution of ξ is anomalously fractal, i.e., it is singular w.r.t. Lebesgue measure and its Hausdorff dimension is equal to zero. We also study properties of the Fourier-Stieltjes transform and coefficients of these measures. In particular, we show that this class of Bernoulli convolutions does not contain any Rajchman measure, i.e., the Fourier-Stieltjes transform of the distribution of ξ does not tend to zero as t tends to infinity. Moreover, in most cases the upper limit of the corresponding sequence of the Fourier-Stieltjes coefficients is equal to 1.

Finite as well as infinite convolutions of distributions of random variables of the form (5) are also studied in details. In particular, we show that any finite such a convolution has an anomalously fractal spectrum. For the limiting case we prove that infinite convolution is of pure type (i.e., it is either purely discretely distributed or absolutely continuously resp. singularly continuously distributed), prove necessary and sufficient conditions for the discreteness and show that the Hausdorff dimension of such an infinite convolution can vary from 0 to 1.

Let $\mu = \mu_\xi$ be the probability measure corresponding to ξ. Since the spectrum S_μ is of zero Hausdorff dimension and, therefore, the Hausdorff dimension of the measure $dim_H(\mu) := \inf\{\dim_H(E), E \in \mathcal{B}, \mu(E) = 1\}$ is also equal to zero, we conclude that the classical Hausdorff dimension does not reflect the difference between the spectrum and other essential supports (see, e.g., [25]) of the singular measure μ. Moreover, if $p'_{0k} = p' \in (0,1)$ and $p''_{0k} = p'' \in (0,1), p'' \neq p'$, then the corresponding random variables ξ' and ξ'' are mutually singularly distributed on the common spectrum S_μ, and all of them are singularly continuous w.r.t. Lebesque measure.

Since the spectra of all random variables ξ are "very poor" in both the metric and the fractal sense (their classical Hausdorff dimensions are equal to zero), to study fine fractal properties of the distribution of ξ it is necessary to apply more delicate tools than $\alpha-$dimensional Hausdorff measure and the corresponding Hausdorff dimension. To this end we consider the so called $h-$Hausdorff measures (see, e.g.,[11]) and Hausdorff-Billingsley dimensions w.r.t. an appropri-

ate probability measure (see, e.g., [7] or Section 6 for details). As an adequate example of such a measure we consider the measure ν^* corresponding to the uniform distribution on the spectrum S_μ. We find necessary and sufficient conditions for μ to be absolutely continuous resp. singular w.r.t. ν^*, prove a formula for the calculation of the Hausdorff–Billingsley dimension of the spectrum of ξ w.r.t. the measure ν^*, and show that the distribution function of the measure ν^* can be considered as a good choice for the gauge function $h(t)$. Moreover, in the same section we study internally fractal properties of ξ. In particular, we find the Hausdorff–Billingsley dimension of the measure μ itself.

In the last section of the paper we develop third approach how to study a level of "irregularity" of probability distributions whose spectra are of zero Hausdorff dimension. We consider a problem of the preservation of the Hausdorff–Billingsley dimension of subsets of the spectrum under the distribution function F_ξ. For the case where elements of the matrix $|p_{ik}|$ are bounded away from zero we find necessary and sufficient conditions for the dimension preservation.

2 Expansions of real numbers via the second Ostrogradsky series.

Definition. The numerical series of following form

$$\frac{1}{q_1} - \frac{1}{q_2} + \frac{1}{q_3} + \ldots + \frac{1}{q_k} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k}, \tag{6}$$

where $\{q_k\}$ is a sequence of positive integers with

$$q_{k+1} \geq q_k(q_k + 1) \tag{7}$$

is said to be the second Ostrogradsky series.

For any irrational number $x \in [0,1]$ there exists a unique sequence $\{q_k = q_k(x)\}$ such that

$$x = \frac{1}{q_1(x)} - \frac{1}{q_2(x)} + \frac{1}{q_3(x)} + \ldots + \frac{1}{q_k(x)} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k(x)}. \tag{8}$$

The sequence $\{q_k\}$ can be determined via the following algorithm:

$$\begin{cases} 1 = q_1 x + \beta_1 \quad (0 \leq \beta_1 < x), \\ q_1 = q_2\beta_1 + \beta_2 \quad (0 \leq \beta_2 < \beta_1), \\ q_2 q_1 = q_3\beta_2 + \beta_3 \quad (0 \leq \beta_3 < \beta_2), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ q_k \ldots q_2 q_1 = q_{k+1}\beta_k + \beta_{k+1} \quad (0 \leq \beta_{k+1} < \beta_k), \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{9}$$

Let us consider several examples of the second Ostrogradsky series:

$$1)\ \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} - \frac{1}{6\cdot 7} + \frac{1}{42\cdot 43} - \frac{1}{1806\cdot 1807} + \dots.$$

Here $q_1 = 1$, and $q_{k+1} = q_k(q_k + 1), \forall k \in N$.

$$2)\ \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^7} - \frac{1}{s^{15}} + \frac{1}{s^{31}} - \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{s^{m_k}},$$

where $2 \le s \in N$, $m_1 = 1$, $m_k = 2m_{k-1} + 1$;

$$3)\ \frac{1}{2\cdot 3} - \frac{1}{2^4\cdot 3} + \frac{1}{2^8\cdot 3^8} - \frac{1}{2^8\cdot 3^{32}} + \frac{1}{2^{64}\cdot 3^{64}} - \frac{1}{2^{256}\cdot 3^{64}} + \dots;$$

$$4)\ \frac{1}{2} - \frac{1}{7} + \frac{1}{59} - \frac{1}{3541} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{p_k},$$

where $\{p_k\}$ is an infinite sequence of prime numbers such that

$$p_1 = 2, \quad p_2 = 7, \quad p_3 = 59, \ \dots,$$

p_k is the minimal prime number such that $p_k \ge p_{k-1}(p_{k-1} + 1)$.

Let us mention some evident properties of denominators of the second Ostrogradsky series.

1. $\dfrac{q_n}{q_{n+1}} \le \frac{1}{q_n+1} < \frac{1}{q_n}$;

2. $\lim\limits_{n\to\infty} \dfrac{q_n}{q_{n+1}} = 0$;

3. $q_n \ge n!$;

4. $\begin{cases} q_2 \ge q_1(q_1+1) > q_1^2; \\ q_3 > q_2^2 > q_1^4; \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q_{n+1} > q_n^2 > q_{n-1}^4 > q_{n-2}^8 > \dots > q_{n-k}^{2^{k+1}} > \dots > q_2^{2^{n-1}} > q_1^{2^n}; \end{cases}$

5. $q_{n+1} > {q_2}^{2^{n-1}} \ge 2^{2^{n-1}}$.

6. $\displaystyle\sum_{i=n+1}^{\infty} \frac{1}{q_i} < \frac{2}{q_{n+1}}$,

   because $\sum\limits_{i=n+1}^{\infty} \frac{1}{q_i} \le \frac{1}{q_{n+1}} + \sum\limits_{i=1}^{\infty} \frac{1}{q_{n+i}(q_{n+i}+1)} < \frac{1}{q_{n+1}} + \frac{1}{q_{n+1}} \sum\limits_{i=1}^{\infty} \frac{1}{q_{n+i}+1} \le$

   $\le \frac{1}{q_{n+1}} \left(1 + \sum\limits_{i=1}^{\infty} \frac{1}{q_{n+i}}\right) < \frac{1}{q_{n+1}} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right) = \frac{2}{q_{n+1}}$.

7. $\dfrac{q_1 \cdots q_n}{q_{n+1}} \leq \dfrac{q_1}{q_2 + 1 + \dfrac{1}{q_2} + \dfrac{1}{q_2 q_3} + \ldots + \dfrac{1}{q_2 \cdots q_{n-1}}} < \dfrac{q_1}{q_2 + 1 + \dfrac{1}{q_2}} < \frac{2}{7};$

$$\frac{q_1 \cdots q_n}{q_{n+1}} \leq \frac{1}{q_1 + 1 + \dfrac{1}{q_1} + \dfrac{1}{q_1 q_2} + \ldots + \dfrac{1}{q_1 \cdots q_{n-1}}} < \frac{1}{q_1 + 1}.$$

8. If $a_n := \frac{1}{q_n}$ and $r_n := \sum\limits_{i=n+1}^{\infty} \dfrac{1}{q_i}$, then $a_n > r_n, \forall n \in N$,

   because $\frac{a_n}{r_n} > \frac{\frac{1}{q_n}}{\frac{2}{q_{n+1}}} = \frac{q_{n+1}}{2q_n} \geq \frac{q_n(q_n+1)}{2q_n} = \frac{q_n+1}{2} \geq 1.$

3 Probability distributions generated by the second Ostrogradsky series and their convolutions

Let r be an irrational number from the unit interval and let $\{q_k = q_k(r)\}$ be the the second Ostrogradsky sequence corresponding to the number r, i.e., $\{q_k\}$ is a unique infinite sequence of positive integers satisfying condition (7), and such that

$$r = \frac{1}{q_1(r)} - \frac{1}{q_2(r)} + \frac{1}{q_3(r)} + \ldots + \frac{1}{q_k(r)} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{q_k(r)}.$$

Let

$$L = \{e : e = (e_1, e_2, \ldots, e_k, \ldots), e_k \in \{0, 1\}\}.$$

The sum $s = s(\{e_k\})$ of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} e_k}{q_k}, \quad \text{where } \{e_k\} \in L, \tag{10}$$

is said to be an *incomplete sum of the series (6)*. It is clear that s depends on the whole *infinite* sequence $\{e_k\}$. We denote the expression (10) and its sum s formally by $\Delta_{e_1 e_2 \ldots e_k \ldots}$. Any partial sum of the series (6) is its incomplete sum. In a very special case where $e_k = 1$, $\forall k \in N$, we obtain the "complete" sum.

Let C_r be the set of all incomplete sums of the series (6). For any $s \in C_r$ there exists a sequence $\{e_k\} = \{e_k(s)\} \in L$ such that

$$s = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} e_k(s)}{q_k}.$$

Let $c_1, c_2, \ldots, c_m$ be a fixed sequence consisting of zeroes and ones. The set $\Delta'_{c_1 c_2 \ldots c_m}$ of all incomplete sums $\Delta_{c_1 c_2 \ldots c_m e_{m+1} \ldots e_{m+k} \ldots}$, where $e_{m+j} \in \{0, 1\}$ for any $j \in N$, is called the *cylinder* of rank m with base $c_1 c_2 \ldots c_m$. It is evident that

$$\Delta'_{c_1 c_2 \ldots c_m a_{m+1}} \subset \Delta'_{c_1 c_2 \ldots c_m}, \quad \forall a_{m+1} \in \{0, 1\}.$$

The closed interval

$$\left[s_m - \sum_{i:2i>m} \frac{1}{q_{2i}}, s_m + \sum_{i:2i-1>m} \frac{1}{q_{2i-1}}\right] \text{with } s_m := \sum_{k=1}^{m} \frac{(-1)^{k-1} c_k}{q_k}$$

is said to be the *cylindrical interval of rank m with base* $c_1 c_2 \dots c_m$ ($c_i \in \{0, 1\}$). We denote it symbolically by $\Delta_{c_1 c_2 \dots c_m}$.

Lemma 1. *The cylindrical intervals have the following properties:*

1. $\Delta'_{c_1 c_2 \dots c_m} \subset \Delta_{c_1 c_2 \dots c_m}$.
2. $\Delta_{c_1 c_2 \dots c_m} = \Delta_{c_1 c_2 \dots c_m 0} \bigcup \Delta_{c_1 c_2 \dots c_m 1}$.
3. *The length of* $\Delta_{c_1 c_2 \dots c_m}$ *is equal to*

$$|\Delta_{c_1 c_2 \dots c_m}| = \operatorname{diam} \Delta'_{c_1 c_2 \dots c_m} = \sum_{k=m+1}^{\infty} \frac{1}{q_k} < \frac{2}{q_{m+1}} \to 0 \quad (m \to \infty).$$

4. $\Delta_{c_1 c_2 \dots c_m 0} \cap \Delta_{c_1 c_2 \dots c_m 1} = \varnothing$.
5. $\bigcap_{m=1}^{\infty} \Delta_{c_1 c_2 \dots c_m} = \bigcap_{m=1}^{\infty} \Delta'_{c_1 c_2 \dots c_m} = s =: \Delta_{c_1 c_2 \dots c_m \dots}$ *for any* $\{c_k\} \in L$.
6. $C_r = \bigcap_{m=1}^{\infty} \bigcup_{c_i \in \{0,1\}} \Delta_{c_1 c_2 \dots c_m}$.

As it has been mentioned above, for the any irrational $r \in [0, 1]$ there exists the sequence $\{q_k\}$: $x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k}$. The sequence $\{q_k\}$ generates the random variable of the following form:

$$\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \xi_k}{q_k}. \tag{11}$$

This random variable can be represented as a "shifted" infinite Bernoulli convolution (see, e.g., [5]), generated by positive convergent series $\sum_{k=1}^{\infty} a_k$ with $a_k = \frac{1}{q_k}$:

$$\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \xi_k}{q_k} = \sum_{k=1}^{\infty} \frac{\eta_k}{q_k} - 2 \sum_{k=1}^{\infty} \frac{1}{q_{2k}} = \sum_{k=1}^{\infty} \frac{\eta_k}{q_k} - const = \eta - const. \tag{12}$$

where η_k also takes values 0 and 1. So, random variables ξ and η have equivalent distributions.

According to the Jessen–Wintner theorem (see, e.g.,[14]) the random variable ξ is of pure type, i.e., it is either pure discrete or pure absolutely continuous resp. pure singularly continuous w.r.t. Lebesgue measure.

The following proposition follows directly from the P. Lévy theorem ([16]) and it gives us necessary and sufficient conditions for the the continuity of ξ.

Lemma 2. *The random variable ξ has a continuous distribution if and only if*

$$D = \prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} = 0.$$

In the sequel we shall be interested in continuous distributions only.

The *spectrum* (*topological support*) S_ξ of the distribution of the random variable ξ is the minimal closed support of ξ. It is clear that $S\mu$ is a perfect set (i.e., a closed set without isolated points). Since

$$S_\xi = \{x : \mathsf{P}\{\xi \in (x-\varepsilon, x+\varepsilon)\} > 0 \ \forall \varepsilon > 0\} = \{x : F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon) > 0 \ \forall \varepsilon > 0\},$$

where F_ξ is the distribution function of the random variable ξ, we deduce that

$$S_\xi = \{x : x \text{ can be represented in the form } \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e_k}{q_k},\ e_k \in \ \{0,1\}\text{with} p_{e_k k} > 0\}.$$

Theorem 1. *The spectrum S_ξ of the distribution of the random variable ξ is a nowhere dense set of zero Hausdorff dimension.*

Proof. From (12) it follows that topological and fractal properties of random variables ξ and η are the same. So, we can apply general results on Bernoulli convolutions (see, e.g., [5]). Since the sequence $\{q_k\}$ is strictly decreasing and $a_k = \frac{1}{q_k} > r_k = \sum\limits_{i=k+1}^{\infty} a_k = \sum\limits_{i=k+1}^{\infty} \frac{1}{q_i}$, we conclude that S_ξ is a nowhere dense set and its Hausdorff dimension can be calculated by the following formula:

$$\dim_H(S_\xi) = \liminf_{k\to\infty} \frac{k \ln 2}{-\ln r_k}.$$

From properties (5) and (6) of denominators of the second Ostrogradsky series it follows that $r_k = \frac{1}{q_{k+1}} + \frac{1}{q_{k+2}} + \frac{1}{q_{k+3}} + \ldots < \frac{2}{q_{k+1}} < \frac{2}{2^{2^{k-1}}}$. So,

$$\dim_H(S_\xi) = \liminf_{k\to\infty} \frac{k \ln 2}{-\ln r_k} \le \lim_{k\to\infty} \frac{k \ln 2}{-\ln \frac{2}{2^{2^{k-1}}}} = 0,$$

which proves the theorem. □

Corollary 1. *If $D = 0$, then ξ has a singularly continuous distribution of the Cantor type with an anomalously fractal spectrum.*

4 Properties of Fourier–Stieltjes transorm of probability distributions generated by the second Ostrogradsky series

Let us consider the *characteristic function $f_\xi(t)$ of the random variable* ξ (Fourier–Stieltjes transorm of the corresponding probability measure), i.e.,

$$f_\xi(t) = \mathsf{E}\left(e^{it\xi}\right).$$

It is well known that for a singularly continuous distribution with the distribution function $F(x)$ one has:

$$\sum_{m=1}^{\infty} {c_m}^2 = \infty,$$

where c_m are the Fourier-Stieltjes coefficients of $F(x)$, i.e.,

$$c_m = \int_{-\infty}^{\infty} e^{2\pi mix} dF(x) = f_\xi(2\pi m).$$

Nevertheless for some classes of singular measures c_m can tend to 0 like for absolutely continuous distributions.

Lemma 3. *The characteristic function of random variable ξ defined by (11) is of the following form*

$$f_\xi(t) = \prod_{k=1}^{\infty} f_k(t) \quad \textit{with} \quad f_k(t) = p_{0k} + p_{1k} \exp \frac{(-1)^{k-1} it}{q_k}, \tag{13}$$

and its absolute value is equal to

$$|f_\xi(t)| = \prod_{k=1}^{\infty} |f_k(t)|, \quad \textit{where} \quad |f_k(t)| = \sqrt{1 - 4p_{0k}p_{1k} \sin^2 \frac{t}{2q_k}}.$$

Proof. Using the definition of a characteristic function and properties of expectations, we have

$$\begin{aligned}
f_\xi(t) = \mathsf{E}\left(e^{it\xi}\right) &= \mathsf{E}\left(\exp\left(it \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\xi_k}{q_k}\right)\right) = \\
&= \mathsf{E}\left(\exp \frac{it\xi_1}{q_1} \cdot \exp \frac{-it\xi_2}{q_2} \cdot \dots \cdot \exp \frac{(-1)^{k-1} it\xi_k}{q_k} \cdot \dots\right) = \\
&= \prod_{k=1}^{\infty} \mathsf{E}\left(\exp \frac{(-1)^{k-1} it\xi_k}{q_k}\right) = \prod_{k=1}^{\infty} \left(p_{0k} + p_{1k} \exp \frac{(-1)^{k-1} it}{q_k}\right) = \\
&= \prod_{k=1}^{\infty} \left((p_{0k} + p_{1k} \cos \frac{(-1)^{k-1} t}{q_k}) + i(p_{1k} \sin \frac{(-1)^{k-1} t}{q_k})\right) = \prod_{k=1}^{\infty} f_k(t),
\end{aligned}$$

and

$$|f_k(t)| = \sqrt{p_{0k}^2 + 2p_{0k}p_{1k} \cos \frac{t}{q_k} + p_{1k}^2} = \sqrt{1 - 4p_{0k}p_{1k} \sin^2 \frac{t}{2q_k}},$$

which proves the lemma. □

Corollary 2. *The Fourier-Stieltjes coefficients of the distribution function of the random variable ξ defined by equality (11) are of the following form*

$$c_m = \prod_{k=1}^{\infty} \left(p_{ok} \; + p_{1k} \exp \frac{(-1)^{k-1} 2\pi m i}{q_k} \right).$$

Let $l.c.m.(m_1, m_2, ..., m_k) := \min\limits_{M \in N} \{M : M \vdots m_1, M \vdots m_2, ..., M \vdots m_k\}$.

Theorem 2. *For any sequence $\{q_n\}$ generated by the second Ostrogradsky series, the Fourier-Stieltjes coefficients of the distribution function of the random variable ξ have the following properties:*

1) $\limsup\limits_{k\to\infty} |c_k| > 0$;

2) *if*

$$\liminf_{k\to\infty} \frac{l.c.m.(q_1, q_2, \dots q_n)}{q_{n+1}} = 0, \tag{14}$$

then $\limsup\limits_{k\to\infty} |c_k| = 1$;

Proof. From the definition follows that $c_k = f_\xi(2\pi k)$.

1.If $k_n = l.c.m.(q_1, q_2, \dots, q_n)$, then $c_{k_n} = f_\xi(l.c.m.(q_1, q_2, \dots, q_n)2\pi)$. Let us estimate

$$|c_{k_n}| = |f_\xi(2\pi k_n)| = \prod_{k=1}^{\infty} \sqrt{1 - 4p_{0k}p_{1k} \sin^2 \frac{\pi k_n}{q_k}} \geq \prod_{k=1}^{\infty} \sqrt{1 - \sin^2 \frac{\pi k_n}{q_k}} =$$

$$= \prod_{k=1}^{\infty} \sqrt{1 - \sin^2 \frac{l.c.m.(q_1, q_2, \dots, q_n)\pi}{q_k}} = \sqrt{\prod_{k=1}^{\infty} 1 - \sin^2 \frac{l.c.m.(q_1, q_2, \dots, q_n)\pi}{q_k}} \geq$$

$$\geq \prod_{k=1}^{\infty} \left(1 - \sin^2 \frac{l.c.m.(q_1, \dots, q_n)\pi}{q_k}\right) = 1 \cdot \ldots \cdot 1 \cdot \prod_{k=n+1}^{\infty} \left(1 - \sin^2 \frac{l.c.m.(q_1, \dots, q_n)\pi}{q_k}\right) \geq$$

$$\geq \prod_{k=n+1}^{\infty} \left(1 - \sin \frac{l.c.m.(q_1, q_2, \dots, q_n)\pi}{q_k}\right) \geq \prod_{k=n+1}^{\infty} \left(1 - \frac{l.c.m.(q_1, q_2, \dots, q_n)\pi}{q_k}\right) = A.$$

Since

$$\frac{q_1}{q_2 + 1 + \frac{1}{q_2}} \leq \frac{2}{7}, \quad \forall\; q_1 \in N; \;\; q_2 \in N, \text{with } q_2 \geq q_1(q_1+1),$$

we have

$$\frac{l.c.m.(q_1, q_2, \dots, q_n)\pi}{q_{n+1}} \leq \frac{q_1 \cdot q_2 \cdot \ldots \cdot q_n \cdot \pi}{q_{n+1}} \leq \frac{q_1 \pi}{q_2 + 1 + \frac{1}{q_2}} \leq \frac{2\pi}{7}.$$

It is well known that

$$(1-b_1)(1-b_2)\cdot\ldots\cdot(1-b_k)\cdot ... \geq 1-(b_1+\ldots+b_k+...)$$

for any sequence $\{b_k\}$ with $0 < b_k < 1$. So,

$$\prod_{k=n+2}^{\infty}\left(1-\frac{l.c.m.(q_1,q_2,\ldots,q_n)\pi}{q_k}\right) \geq 1-\sum_{k=n+2}^{\infty}\left(\frac{l.c.m.(q_1,q_2,\ldots,q_n)\pi}{q_k}\right) =$$

$$= 1-\sum_{k=n+2}^{\infty}\frac{l.c.m.(q_1,q_2,\ldots,q_n)\pi}{q_{n+1}}\cdot\frac{q_{n+1}}{q_k} \geq 1-\frac{\pi}{3}\cdot\sum_{k=n+2}^{\infty}\frac{q_{n+1}}{q_k} = B.$$

Since

$$\frac{q_{n+1}}{q_{n+2}} \leq \frac{1}{q_{n+1}} < \frac{1}{2^{2^{n-1}}};$$

$$\frac{q_{n+1}}{q_{n+m}} = \frac{q_{n+1}}{q_{n+2}}\cdot\frac{q_{n+2}}{q_{n+3}}\cdot\ldots\cdot\frac{q_{n+m-1}}{q_{n+m}} < \frac{1}{2^{2^{n-1}}}\cdot\frac{1}{2^{2^n}}\cdot\ldots\cdot\frac{1}{2^{2^{n+m-3}}} \leq \frac{1}{2^{2^{n-1}}}\cdot\frac{1}{2^{m-2}},$$

we get

$$B \geq 1-\frac{\pi}{3}\cdot\frac{1}{2^{2^{n-1}}}\left(1+\frac{1}{2}+\frac{1}{4}+\ldots\right) = 1-\frac{2\pi}{3\cdot 2^{2^{n-1}}}.$$

Therefore,

$$A \geq \left(1-\frac{2\pi}{7}\right)\left(1-\frac{2\pi}{3\cdot 2^{2^{n-1}}}\right) > \left(1-\frac{2\pi}{7}\right)\left(1-\frac{\pi}{6}\right), \quad \forall\ n \in N.$$

Hence,

$$\limsup_{k\to\infty}|c_k| \geq \limsup_{n\to\infty}|c_{k_n}| \geq \left(1-\frac{2\pi}{7}\right)\left(1-\frac{\pi}{6}\right) > 0\ .$$

2. If condition (2) holds, then there exists a sequence $\{n_s\}$ of positive integers such that

$$\liminf_{n\to\infty}\frac{l.c.m.(q_1,q_2,\ldots q_{n_s})}{q_{n_s+1}} = 0.$$

Let us consider the sequence $k_{n_s} = l.c.m.(q_1,q_2,\ldots,q_{n_s})$. Using our previous arguments, we have

$$|c_{k_{n_s}}| \geq \left(1-\frac{l.c.m.(q_1,q_2,\ldots,q_{n_s})}{q_{n_s+1}}\right)\cdot\left(1-\frac{\pi}{3}\cdot\frac{2}{2^{2^{n_s-1}}}\right) \to 1 \quad (s\to\infty).$$

Therefore,

$$\limsup_{k\to\infty}|c_k| = 1.$$

$\square$

For a given random variable ζ one can define

$$L_\xi = \limsup_{|t|\to\infty} |f_\xi(t)| .$$

It is well known that $L_\zeta = 1$ for any discretely distributed random variable ζ. If ζ has an absolutely continuous distribution, then $L_\zeta = 0$. If ζ has a singularly continuous distribution, then $L_\zeta \in [0,1]$. More precisely: for any real number $\beta \in [0,1]$ there exists a singularly distributed random variable ζ_β such that $L_{\zeta_\beta} = \beta$. Let us stress the asymptotic behaviour at infinity of the absolute value of the characteristic function of the random variable ξ,defined by (11).

Corollary 3. *For any sequence $\{q_n\}$ generated by the second Ostrogradsky series, we have*

$$L_\xi > 0.$$

If $\liminf\limits_{k\to\infty} \frac{l.c.m.(q_1,q_2,\dots q_n)}{q_{n+1}} = 0,$ *then* $L_\xi = 1.$

Proof. It is clear that

$$L_\xi = \limsup_{|t|\to\infty} |f_\xi(t)| \geq \limsup_{k\to\infty} |f_\xi(2\pi k)| = \limsup_{k\to\infty} |c_k|,$$

and the statement of corollary follows directly from statements 1) and 2) of the latter theorem. ☐

5 Convolutions of singular distributions generated by the second Ostrogradsky series

It is well known (see, e.g., [17]) that for a random variable ζ which is a sum of two independent random variables ζ_1 and ζ_2 one has $F_\zeta = F_{\zeta_1} * F_{\zeta_2}$ and $f_\zeta = f_{\zeta_1} \cdot f_{\zeta_2}$, where F_ζ resp. f_ζ means the probability distribution function resp. characteristic function of the random variable ζ. If either ζ_1 or ζ_2 has an absolutely continuous distribution then ζ also has a density. If both ζ_1 and ζ_2 has singular distribution then, generally speaking, nothing known about the distribution of ζ. The convolution of two singular probability distributions is either singular or absolutely continuous, or is of a mixed type. An absolutely continuous distribution can arise even as a convolution of two anomalously fractal singularly continuous distributions. The desirability of finding a criterium for the singularity resp. absolute continuity of the convolution of two singular distributions has been expressed by many authors (see, e.g., [17]), but it is still unknown. It can however be given for special classes of random variables (see, e.g., [2] and references therein).

In this Section we study finite as well as infinite convolutions of probability distributions generated by the second Ostrogradsky series.

5.1 Autoconvolutions

Theorem 3. *Let*

$$\xi^{(j)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot \xi_k^{(j)}}{q_k},$$

where $\{q_k\}$ is a sequence of positive integers, $q_{k+1} \geq q_k(q_k+1)$, let $\{\xi_k^{(j)}\}$ be sequences of random variables taking values 0 and 1 with probabilities p_{0k} and p_{1k} correspondingly. Let $\{\xi^{(j)}\}$ be mutually independent random variables and

$$\psi = \sum_{j=1}^{m} \xi^{(j)}.$$

Then the random variable ψ has a singular distribution with

$$\dim_H (S_\psi) = 0$$

and

$$L_\psi = \limsup_{|t|\to\infty} |f_\psi(t)| > 0.$$

Proof. If $\xi_1^{(j)} = i_1^{(j)},\ \xi_2^{(j)} = i_2^{(j)}, \ldots, \xi_n^{(j)} = i_n^{(j)},\ \ i_s^{(j)} \in \{0,1\},\ s = \overline{1,n}, j = \overline{1,m}$, then

$$\sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(1)}}{q_s} + \ldots + \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(m)}}{q_s} - \left[\sum_{s=n+1}^{\infty} \frac{1}{q_s} + \ldots + \sum_{s=n+1}^{\infty} \frac{1}{q_s}\right] \leq$$

$$\leq \xi^{(1)} + \xi^{(2)} + \ldots + \xi^{(m)} \leq$$

$$\leq \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(1)}}{q_s} + \ldots + \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(m)}}{q_s} + \left[\sum_{s=n+1}^{\infty} \frac{1}{q_s} + \ldots + \sum_{s=n+1}^{\infty} \frac{1}{q_s}\right]$$

Taking into account that $\sum\limits_{s=n+1}^{\infty} \frac{1}{q_s} < \frac{2}{q_{n+1}}$ (see property 6 of denominators of the Ostrogradsky series), we conclude that

$$\sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(1)}}{q_s} + \ldots + \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(m)}}{q_s} - m \cdot \frac{2}{q_{n+1}} \leq \psi \leq$$

$$\leq \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(1)}}{q_s} + \ldots + \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(m)}}{q_s} + m \cdot \frac{2}{q_{n+1}}$$

The random variable $\xi_k^{(1)}+\xi_k^{(2)}+\ldots+\xi_k^{(m)}$ takes values from the set $\{0,1,\ldots,m\}$. So, for any $n \in N$ the spectrum S_ψ of the random variable ψ can be covered by $(m+1)^n$ intervals of length $\frac{4m}{q_{n+1}}$. The $\alpha-$ volume of this covering is equal to

$$(m+1)^n \cdot \left(\frac{4m}{q_{n+1}}\right)^{\alpha} = (4m)^{\alpha} \cdot \frac{(m+1)^n}{{q_{n+1}}^{\alpha}} \leq (4m)^{\alpha} \cdot \frac{(m+1)^n}{(2^{2^{n-2}})^{\alpha}} \to 0 \ (n \to \infty),\ \forall m \in \mathbb{N}.$$

So,

$$H^{\alpha}(S_{\psi}) = 0, \quad \forall\ \alpha > 0.$$

Hence,

$$\dim_H(S_{\psi}) = \inf\{\alpha:\ H^{\alpha}(S_{\psi}) = 0\} = 0.$$

Since the random variable $\xi^{(1)}, \xi^{(2)}, ..., \xi^{(m)}$ are mutually independent, from general properties of the Fourier-Stieltjes transform it follows that

$$f_{\psi}(t) = \prod_{j=1}^{m} f_{\xi}^{(j)}(t) = \left(f_{\xi}^{(j)}\right)^{m} = \left(f_{\xi}(t)\right)^{m}.$$

Applying results of theorem 2, we get the desired statement:

$$\limsup_{|t|\to 0} |f_{\psi}(t)| = \limsup_{|t|\to 0} |f_{\xi}(t)|^{m} = \left(\limsup_{|t|\to 0} |f_{\xi}(t)|\right)^{m} > 0.$$

□

Remark. 1) From the proof of the latter theorem it follows that the random variable $\psi = \xi^{(1)} + ... + \xi^{(k)}$ has an anomalously fractal singular distribution even in the case where random variables $\xi^{(1)}, ..., \xi^{(k)}$ are not independent.

2) It is impossible to consider infinite autoconvolutions, because the resulting random series will diverge almost surely.

5.2 General convolutions of singular distributions generated by the second Ostrogradsky series

Let $\{q_k^{(j)}\}$ be sequences of positive integers, $q_{k+1}^{(j)} \geq q_k^{(j)}(q_k^{(j)} + 1)$, let $\{\xi_k^{(j)}\}$ be sequences of random variables taking values 0 and 1 with probabilities $p_{0k}^{(j)}$ and $p_{1k}^{(j)}$ correspondingly, and let

$$\xi^{(j)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot \xi_k^{(j)}}{q_k^{(j)}}$$

.

Theorem 4. *If*

$$\widetilde{\psi} = \sum_{j=1}^{m} \xi^{(j)}, \tag{15}$$

then the random variable $\widetilde{\psi}$ has an anomalously fractal singular distribution.

Proof. If $\xi_1^{(j)} = i_1^{(j)},\ \xi_2^{(j)} = i_2^{(j)}, \ldots, \xi_n^{(j)} = i_n^{(j)},\ \ i_s^{(j)} \in \{0,1\},\ s = \overline{1,n}, j = \overline{1,k}$, then

$$\sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(1)}}{q_s^{(1)}} + \ldots + \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(k)}}{q_s^{(m)}} - \left[\sum_{s=n+1}^{\infty} \frac{1}{q_s^{(1)}} + \ldots + \sum_{s=n+1}^{\infty} \frac{1}{q_s^{(m)}} \right] \leq$$

$$\leq \xi^{(1)} + \xi^{(2)} + \ldots + \xi^{(k)} \leq$$

$$\leq \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(1)}}{q_s^{(1)}} + \ldots + \sum_{s=1}^{n} \frac{(-1)^{s+1} \cdot i_s^{(k)}}{q_s^{(m)}} + \left[\sum_{s=n+1}^{\infty} \frac{1}{q_s^{(1)}} + \ldots + \sum_{s=n+1}^{\infty} \frac{1}{q_s^{(m)}} \right].$$

It is clear that

$$\sum_{s=n+1}^{\infty} \frac{1}{q_s^{(1)}} + \ldots + \sum_{s=n+1}^{\infty} \frac{1}{q_s^{(m)}} \leq m \cdot \max\{ \sum_{s=n+1}^{\infty} \frac{1}{q_s^{(1)}}, ..., \sum_{s=n+1}^{\infty} \frac{1}{q_s^{(m)}} \} \leq$$

$$m \cdot \max\{ \frac{2}{q_{n+1}^{(1)}}, ..., \frac{2}{q_{n+1}^{(m)}} \} \leq m \cdot \frac{2}{2^{2^{n-1}}}.$$

Since the random variable $\frac{\xi_k^{(1)}}{q_k^{(1)}} + \ldots + \frac{\xi_k^{(m)}}{q_k^{(m)}}$ takes at most 2^m values, we conclude that the spectrum $S_{\widetilde{\psi}}$ of the random variable $\widetilde{\psi}$ can be covered by 2^{mn} intervals of length $m \cdot \frac{4}{2^{2^{n-1}}}$. The $\alpha-$ volume of this covering is equal to

$$2^{mn} \cdot \left(\frac{4m}{2^{2^{n-1}}} \right)^{\alpha} \to 0 \ \ (n \to \infty), \quad \forall \alpha > 0,\ \forall m \in \mathbb{N}.$$

So, the Hausdorff measure $H^{\alpha}(S_{\widetilde{\psi}})$ of the spectrum of $\widetilde{\psi}$ is equal to zero for any positive α. Therefore, $\dim_H (S_{\psi}) = 0$. □

Let us now consider the case where

$$\sum_{j=1}^{\infty} \frac{1}{q_k^{(j)}} < +\infty. \tag{16}$$

In such a case the random variable

$$\widetilde{\psi}_{\infty} = \sum_{j=1}^{\infty} \xi^{(j)} = \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \xi_k^{(j)}}{q_1^{(j)}} \right) \tag{17}$$

is correctly defined and it has a bounded spectrum: $S_{\widetilde{\psi}_{\infty}} \subset \left[- \sum\limits_{j=1}^{\infty} \frac{1}{q_1^{(j)}},\ \sum\limits_{j=1}^{\infty} \frac{1}{q_1^{(j)}} \right]$.

Theorem 5. *The random variable $\widetilde{\psi}_\infty$ is of pure type, i.e., it is either purely discretely distributed or purely singularly continuously resp. purely absolutely continuously distributed. It has a pure discrete distribution if and only if*

$$\prod_{j=1}^{\infty}\left(\prod_{k=1}^{\infty}\max_k\{p_{0k}^{(j)},p_{1k}^{(j)}\}\right)>0. \tag{18}$$

Proof. From property 6 of denominators of the second Ostrogradsky series and condition 16 follows that $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{q_k^{(j)}}\leq\sum_{j=1}^{\infty}\frac{2}{q_1^{(j)}}<+\infty$. So, the random variable $\widetilde{\psi}_\infty$ can be represented in the following form:

$$\widetilde{\psi}_\infty=\sum_{d=1}^{\infty}\left(\sum_{k+j=d}^{\infty}\frac{(-1)^{k+1}\xi_k^{(j)}}{q_k^{(j)}}\right), \tag{19}$$

i.e., $\widetilde{\psi}_\infty$ can be represented as a sum of convergent series of independent discretely distributed random variables, and, therefore, from the Jessen-Wintner theorem it follows that $\widetilde{\psi}_\infty$ has the distribution of pure type.

The atom of maximal weight of the distribution of $\xi^{(j)}$ is equal to $D_j:=\prod_{k=1}^{\infty}\max_k\{p_{0k}^{(j)},p_{1k}^{(j)}\}$. So, from Lévi theorem ([16]) it follows that $\widetilde{\psi}_\infty$ is purely discretely distributed if and only if the product $\prod_{j=1}^{\infty}D_j$ converges, which prove the theorem. □

Remark. If for some $j\in N$ the random variable $\xi^{(j)}$ has a continuous distribution, then $\widetilde{\psi}_\infty$ also has no atoms. But it can happens that $\xi^{(j)}$ is pure atomic for all $j\in N$, but $\widetilde{\psi}_\infty$ does not.

As has been shown before, the sum $\sum_{j=1}^{m}\xi^{(j)}$ is always singular and the corresponding spectrum is of zero Hausdorff dimension. In the limiting case the situation is more complicated. The random variable $\widetilde{\psi}_\infty$ can be absolutely continuous as well as singularly continuous with different values of the Hausdorff dimension of the spectrum. Let us consider two "critical" cases.

Example 1. Let $q_1^{(j)}=\frac{1}{2^j}$, $p_{01}^{(j)}=\frac{1}{2}$. In such a case

$$\widetilde{\psi}_\infty=\sum_{j=1}^{\infty}\frac{\xi_1^{(j)}}{2^j}+\sum_{j=1}^{\infty}\left(\sum_{k=2}^{\infty}\frac{(-1)^{k+1}\xi_k^{(j)}}{q_k^{(j)}}\right).$$

Since the random variable $\sum_{j=1}^{\infty}\frac{\xi_1^{(j)}}{2^j}$ has uniform distribution on the unit interval, we get absolute continuity of the distribution of $\widetilde{\psi}_\infty$. In this case $\dim_H(S_{\widetilde{\psi}_\infty})=1$.

Example 2. Let $\{q_k\}$ be an arbitrary Ostrogradsky sequence of positive integers. Let $q_k^{(j)}=\frac{1}{q_{k+j}}$. In such a case the random variable $\widetilde{\psi}_\infty$ has a singular distribution with anomalously fractal spectrum for any choice of probabilities $p_{ik}^{(j)}$.

6 Fine fractal properties of the distribution of the random variable ξ

The main aim of this section is the study fine fractal properties of the distribution of the random variable ξ generated by a given Ostrogradsky sequence (see (11) to remind the exact definition). Let $\mu = \mu_\xi$ be the probability measure corresponding to ξ. The spectrum S_μ has zero Hausdorff dimension and, therefore, the Hausdorff dimension of the measure $dim_H(\mu) := \inf\{\dim_H(E), E \in \mathcal{B}, \mu(E) = 1\}$ is also equal to zero. So, in such a case the classical Hausdorff dimension does not reflect the difference between the spectrum and other essential supports (see, e.g., [25]) of the singular measure μ.

So, study fine fractal properties of the distribution of ξ it is necessary to apply more delicate tools than $\alpha-$dimensional Hausdorff measure and the corresponding Hausdorff dimension. To this end let us consider the so called $h-$Hausdorff measures. Let $h(t) : R_+ \to R_+$ be a continuous increasing (non-decreasing) function such that $\lim\limits_{t\to 0} h(t) = 0$. Usually the function h is called the dimensional function or gauge function. For a given set E, a given gauge function h and a given $\varepsilon > 0$, let

$$H^h_\varepsilon(E) := \inf_{|E_j|\le\varepsilon} \sum h(|E_j|), \bigcup_j E_j \supseteq E,$$

where the infimum is taken over all ε-coverings $\{E_j\}$ of the set E.

Since $H^h_{\varepsilon_1}(E) \ge H^h_{\varepsilon_2}(E)$ for $\varepsilon_1 < \varepsilon_2$, the following limit

$$H^h(E) := \lim_{\varepsilon\to 0} H^h_\varepsilon(E)$$

exists, and is said to be the h-Hausdorff measure (or H^h-measure) of the set E. For a given set E and a given gauge function h the value $H^h(E)$ can be either zero or positive and finite, or to be equal $+\infty$. If $h(t) = t^\alpha$, then we get the classical Hausdorff measure. If h_1 and h_2 are dimension functions such that $\lim\limits_{t\to 0} \frac{h_1(t)}{h_2(t)} = 0$, then $H^{h_1}(E) = 0$ whenever $H^{h_2}(E) < +\infty$ (see, e.g., [11, 27] for details). Thus partitioning the dimension functions into those for which H^h is finite and those for which it is infinite gives a more precise information about fine fractal properties of a set E. An important example of this is Brownian motion in R^3 (see Chapter 16 of [11] for details). It can be shown that almost surely a Brownian path is of the Hausdorff dimension 2, but their H^2-measure is equal to 0. More refined calculations show that such a path has positive and finite H^h-measure, where $h(t) = t^2 \log(\log(\frac{1}{t}))$.

So, our first aim of this section is to find a gauge function h for the spectrum of the random variable ξ and study fractal properties of the probability measure μ_ξ w.r.t. the measure H^h. To this aim let us remind the notion of the Hausdorff–Billingsley dimension.

Let M be a fixed bounded subset of the real line. A family Φ_M of intervals is said to be a *fine covering family* for M if for any subset $E \subset M$, and for any $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_j\}$ of E, $E_j \in \Phi_M$. A fine covering family Φ_M is said to be *fractal* if for the determination of the Hausdorff dimension of any subset $E \subset M$ it is enough to consider only coverings from Φ_M.

For a given bounded subset M of the real line, let Φ_M be a fine covering family for M, let α be a positive number and let ν be a continuous probability measure. The ν-α-Hausdorff–Billingsley measure of a subset $E \subset M$ w.r.t. Φ_M is defined as follows:

$$H^{\alpha}(E, \nu, \Phi_M) = \lim_{\varepsilon \to 0} \left\{ \inf_{\nu(E_j) \leq \varepsilon} \sum_j (\nu(E_j))^{\alpha} \right\},$$

where $E_j \in \Phi_M$ and $\bigcup\limits_j E_j \supset E$.

Definition. The number $\dim_H(E, \nu, \Phi_M) = \inf\{\alpha : H^{\alpha}(E, \nu, \Phi_M) = 0\}$ is called the *Hausdorff–Billingsley dimension of the set E with respect to the measure ν and the family of coverings Φ_M.*

Remark. 1) Let Φ_M be the family of all closed subintervals of the minimal closed interval $[a, b]$ containing M. Then for any $E \subset M$ the number $\dim_H(E, \nu, \Phi_M)$ coincides with the classical Hausdorff–Billingsley dimension $\dim_H(E, \nu)$ of the subset E w.r.t. the measure ν.

2) Let $M = [0, 1]$, ν be the Lebesgue measure on $[0, 1]$ and let Φ_M be a fractal family of coverings. Then for any $E \subset M$ the number $\dim_H(E, \nu, \Phi_M)$ coincides with the classical Hausdorff dimension $\dim_H(E)$ of the subset E.

Let Φ^{ν}_M be the image of a fine covering family under the distribution function of a probability measure ν, i.e., $\Phi^{\nu}_M = \{E' : E' = F_{\nu}(E), E \in \Phi_M\}$. The following lemma has been proven in [1].

Lemma 4. *A fine covering family Φ_M can be used for the equivalent definition of the Hausdorff–Billingsley dimension of any subset $E \subset M$ w.r.t. a measure ν if and only if the covering family Φ^{ν}_M can be used for the equivalent definition of the classical Hausdorff dimension of any subset $E' = F_{\nu}(E)$, $E \subset M$, i.e.,*

$$\dim_H(E, \nu, \Phi_M) = \dim_H(E, \nu) \quad \textit{for any } E \subset M$$

if and only if the covering family Φ^{ν}_M is fractal.

Definition. The number

$$\dim_H(\mu, \nu) = \inf_{E \in B_{\eta}} \{\dim_H(E, \nu), E \in \mathcal{B}, \mu(E) = 1\}$$

is said to be the *Hausdorff–Billingsley dimension of the measure μ with respect to the measure ν.*

To show the difference between the spectrum and essential supports of the measure $\mu = \mu_\xi$ it is natural to use the Hausdorff–Billingsley dimension with respect to the measure ν^*, where ν^* is the probability measure, which is "uniformly distributed" on the spectrum of the measure μ_ξ, i.e., ν^* is the probability measure corresponding to the random variable

$$\xi^* = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\xi_k^*}{q_k}, \tag{20}$$

where ξ_k^* are independent random variables taking the values 0 and 1 with probabilities p_{0k}^* and p_{1k}^* such that for any $i \in \{0,1\}$:
$p_{ik}^* = 0$ if and only if $p_{ik} = 0$;
$p_{ik}^* = 1$ if and only if $p_{ik} = 1$;
$p_{ik}^* = \frac{1}{2}$ if and only if $p_{ik} \in (0,1)$.

Remark. The measure ν^* can be considered as a substitute of the Lebesgue measure on the set S_μ. If $p_{ik} \in (0,1), \forall i \in \{0,1\}, \forall k \in N$, then the measure ν^* is uniformly distributed not only on the spectrum S_μ, but also on the set C_r of all incomplete sums of the second Ostrogradsky series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k}$. But, generally speaking, the spectrum S_μ of the measure μ_ξ can be essentially smaller than the set C_r. It is clear that measures μ and ν^* have a common spectrum.

Theorem 6. *The measure μ is absolutely continuous w.r.t. the measure ν^* if and only if*

$$\sum_{k:\ p_{0k}\cdot p_{1k}>0} (1-2p_{0k})^2 < +\infty; \tag{21}$$

The measure μ is singularly continuous w.r.t. the measure ν^* if and only if $D = \prod\limits_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} = 0$ and

$$\sum_{k:\ p_{0k}\cdot p_{1k}>0} (1-2p_{0k})^2 = +\infty. \tag{22}$$

Proof. Let $\Omega_k = \{0,1\}$, $\mathcal{A}_k = 2^{\Omega_k}$. We define measures μ_k and ν_k in the following way:

$$\mu_k(i) = p_{ik}; \nu_k(i) = p_{ik}^*, \quad i \in \Omega_k.$$

Let

$$(\Omega, \mathcal{A}, \overline{\mu}) = \prod_{k=1}^{\infty}(\Omega_k, \mathcal{A}_k, \mu_k), (\Omega, \mathcal{A}, \nu) = \prod_{k=1}^{\infty}(\Omega_k, \mathcal{A}_k, \nu_k)$$

be the infinite products of probability spaces, and let us consider the measurable mapping $f : \Omega \to R^1$ defined as follows:

$$\forall \omega = (\omega_1, \omega_2, ..., \omega_k, ...) \in \Omega, f(\omega) = x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\omega_k}{q_k}.$$

We define the measures $\widetilde{\mu}$ and $\widetilde{\nu}$ as image measures of $\overline{\mu}$ resp. ν under f:

$$\widetilde{\mu}(B) := \overline{\mu}(f^{-1}(B)); \widetilde{\nu}(B) := \nu(f^{-1}(B)), B \in \mathcal{B}.$$

It is easy to see that $\widetilde{\nu}$ coincides with the measure ν^*, and $\widetilde{\mu}$ coincides with μ. This mapping is bijective and bi-measurable. Therefore (see, e.g., [6]), the measure μ is absolutely continuous (singular) with respect to the measure ν^* if and only if the measure $\widetilde{\mu}$ is absolutely continuous (singular) with respect to the measure $\widetilde{\nu}$. By construction, $\mu_k \ll \nu_k$, $\forall k \in N$. By using Kakutani's theorem [15], we have

$$\mu_\xi \ll \lambda \ \Leftrightarrow \prod_{k=1}^{\infty} \left(\int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k \right) > 0 \ \Leftrightarrow \ \prod_{k=1}^{\infty} \left(\sum_{i \in \Omega_k} \sqrt{p_{ik} q_{ik}} \right) > 0, \tag{23}$$

$$\mu_\xi \perp \lambda \ \Leftrightarrow \ \prod_{k=1}^{\infty} \left(\int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k \right) = 0 \Leftrightarrow \ \prod_{k=1}^{\infty} \left(\sum_{i \in \Omega_k} \sqrt{p_{ik} q_{ik}} \right) = 0. \tag{24}$$

It is clear that $\sum\limits_{i \in \Omega_k} \sqrt{p_{ik} p^*_{ik}} = 1$ if $p_{0k} \cdot p_{1k} = 0$, and $\sum\limits_{i \in \Omega_k} \sqrt{p_{ik} q_{ik}} = \sqrt{\frac{1}{2} p_{0k}} + \sqrt{\frac{1}{2} p_{1k}}$ if $p_{0k} \cdot p_{1k} > 0$. Since $p_{0k} + p_{1k} = 1$, it is not hard to check that the product $\prod\limits_{k=1}^{\infty} \left(\sum\limits_{i \in \Omega_k} \sqrt{p_{ik} q_{ik}} \right) = \prod\limits_{k:\ p_{0k} \cdot p_{1k} > 0} \left(\sqrt{\frac{1}{2} p_{0k}} + \sqrt{\frac{1}{2} p_{1k}} \right)$ converges to a positive constant if and only if the series $\sum\limits_{k:\ p_{0k} \cdot p_{1k} > 0} (1 - 2p_{0k})^2$ converges, which proves the theorem. $\square$

Remark. Since the measures μ and ν^* have a common spectrum, the absolute continuity of μ w.r.t. ν^* means the equivalence of these measures (i.e., $\mu \ll \nu^*$ and $\nu^* \ll \mu$).

If $\max\{p_{0k}, p_{1k}\} = 1$ for all large enough $k \in N$, then μ is discretely distributed with a finite number of atoms. So, in the sequel we shall assume that there are infinitely many indices k such that $\max\{p_{0k}, p_{1k}\} < 1$, which is equivalent to the continuity of the measure ν^*.

Theorem 7. *Let $h_n = -(p_{0n} \ln p_{0n} + p_{1n} \ln p_{1n})$ be the entropy of the random variable ξ_n and let $H_n = h_1 + h_2 + \cdots + h_n$. Then the Hausdorff–Billingsley dimension of the measure μ_ξ with respect to the measure ν^* is equal to*

$$\dim_H(\mu, \nu^*) = \liminf_{n \to \infty} \frac{H_n}{g_n \ln 2},$$

where g_n is the number of positive elements among $h_1, ..., h_n$.

Proof. Let $r = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k}$, let C_r be the set of all incomplete sums of this second Ostrogradsky series and let $\widetilde{\Phi}_{C_r}$ be the family of the above mentioned cylindrical intervals (see Section 3 for details), i.e., $\widetilde{\Phi}_{C_r} = \{\Delta_{c_1c_2\dots c_k}, k \in \mathbb{N}, c_i \in \{0,1\}\}$, and $C_r = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{c_i \in \{0,1\}} \Delta_{c_1c_2\dots c_k}$.

Let $M = S_\mu \subset C_r$, and let Φ_M be the following subfamily of $\widetilde{\Phi}_{C_r}$:

$$\Phi_M = \{\Delta_{c_1c_2\dots c_n}, n \in \mathbb{N};\ \ c_j \in \{0,1\} \text{ if } h_j > 0, \text{ and } c_j = i_j \text{ if } h_j = 0 \text{ with } p_{i_j j} = 1\}.$$

It is clear that Φ_M is a fine covering family for the spectrum S_μ.

Let $\Delta_{c_1c_2\dots c_n} \in \Phi_M$. Since $\nu^*(\Delta_{c_1c_2\dots c_n}) = 2^{-g_n}$, the image $\Phi_M^{\nu^*} = F_{\nu^*}(\Phi_M)$ coincides with the fractal fine covering family consisting of binary closed subintervals of $[0,1]$. So, from Lemma 4 it follows that for the determination of the Hausdorff–Billingsley dimension of an arbitrary subset of S_μ w.r.t. ν^* it is enough to consider only coverings consisting of cylindrical intervals from Φ_M.

Let $\Delta_n(x) = \Delta_{c_1(x)c_2(x)\dots c_n(x)}$ be the cylindrical interval of the n-th rank containing a point x from the spectrum S_μ. Then we have

$$\mu(\Delta_n(x)) = p_{c_1(x)1} \cdot p_{c_2(x)2} \cdot \dots \cdot p_{c_n(x)n}, \quad \nu^*(\Delta_n(x)) = \frac{1}{2^{g_n}}.$$

Let us consider the following expression

$$\frac{\ln \mu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} = \frac{\sum\limits_{j=1}^{n} \ln p_{c_j(x)j}}{-g_n \ln 2}.$$

If $x = \Delta_{c_1(x)c_2(x)\dots c_n(x)\dots} \in S_\mu$ is chosen stochastically such that $\mathsf{P}\{c_j(x) = i\} = p_{ij} > 0$ (i.e., the distribution of the random variable x corresponds to the measure μ), then $\{\eta_j\} = \{\eta_j(x)\} = \{\ln p_{c_j(x)j}\}$ is a sequence of independent random variables taking the values $\ln p_{0j}$ and $\ln p_{1j}$ with probabilities p_{0j} and p_{1j} respectively.

$$\mathsf{E}(\eta_j) = p_{0j} \ln p_{0j} + p_{1j} \ln p_{1j} = -h_j, \quad |h_j| \le \ln 2,$$

$$\mathsf{E}(\eta_j^2) = p_{0j} \ln^2 p_{0j} + p_{1j} \ln^2 p_{1j} \le c_0 < \infty,$$

and the constant c_0 does not depend on j.

If $p_{i_j j} = 1$, then η_j takes the value $0 = E(\eta_j)$ with probability 1. So, in the sum $\eta_1(x) + \eta_2(x) + \dots + \eta_n(x)$ there are at most g_n non-zero addends. It is clear that $g_n \to \infty$ as $n \to \infty$. By using the strong law of large numbers (Kolmogorov's theorem, see, e.g., [23, Chapter III.3.2]), for μ-almost all points $x \in S_\mu$ the following equality holds:

$$\lim_{n \to \infty} \frac{(\eta_1(x) + \eta_2(x) + \dots + \eta_n(x)) - \mathsf{E}(\eta_1(x) + \eta_2(x) + \dots + \eta_n(x))}{g_n} = 0.$$

We remark that $\mathsf{E}(\eta_1 + \eta_2 + \cdots + \eta_n) = -H_n$.

Let us consider the set

$$A = \left\{ x : \lim_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} - \frac{H_n}{g_n \ln 2} \right) = 0 \right\} =$$
$$= \left\{ x : \lim_{n\to\infty} \frac{(\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)) - \mathsf{E}(\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x))}{-g_n \ln 2} = 0 \right\}.$$

Since $\mu(A) = 1$, we have $\dim_H(A, \mu) = 1$, and, therefore, $\dim_H(A, \mu, \Phi_M) = 1$.

Let us consider the following sets

$$A_1 = \left\{ x : \liminf_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} - \frac{H_n}{g_n \ln 2} \right) = 0 \right\};$$
$$A_2 = \left\{ x : \liminf_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} \leq \liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} \right\} =$$
$$= \left\{ x : \liminf_{n\to\infty} \frac{\ln \mu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \leq \liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} \right\};$$
$$A_3 = \left\{ x : \liminf_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} \geq \liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} \right\} =$$
$$= \left\{ x : \liminf_{n\to\infty} \frac{\ln \mu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \geq \liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} \right\}.$$

It is obvious that $A \subset A_1$. We now prove the inclusions $A_1 \subset A_3$ and $A \subset A_2$. To this end we use the well-known inequality

$$\liminf_{n\to\infty}(x_n - y_n) \leq \liminf_{n\to\infty} x_n - \liminf_{n\to\infty} y_n$$

holding for arbitrary sequences $\{x_n\}$ and $\{y_n\}$ of real numbers.

If $x \in A_1$, then

$$\liminf_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} - \liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} \geq$$
$$\geq \liminf_{n\to\infty} \left(\frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} - \frac{H_n}{g_n \ln 2} \right) = 0.$$

So, $x \in A_3$.

If $x \in A$, then

$$\liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} - \liminf_{n\to\infty} \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} \geq$$
$$\geq \liminf_{n\to\infty} \left(\frac{H_n}{g_n \ln 2} - \frac{\eta_1(x) + \eta_2(x) + \cdots + \eta_n(x)}{-g_n \ln 2} \right) = 0.$$

So, $x \in A_2$.

Let $D_0 = \liminf\limits_{n\to\infty} \frac{H_n}{g_n \ln 2}$. Since $A \subset A_2 = \left\{x : \liminf\limits_{n\to\infty} \frac{\ln \mu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \le D_0\right\}$, we have

$$\dim_H(A, \nu^*, \Phi_M) \le D_0.$$

Since $A \subset A_3 = \left\{x : \liminf\limits_{n\to\infty} \frac{\ln \mu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \ge D_0\right\}$, and $\dim_H(A, \mu, \Phi_M) = 1$, we deduce that

$$\dim_H(A, \nu^*, \Phi_M) \ge D_0 \cdot \dim_H(A, \mu, \Phi_M) = D_0$$

Therefore,

$$\dim_H(A, \nu^*, \Phi_M) = \dim_H(A, \nu^*) = D_0.$$

Let us now prove that the above constructed set A is the "smallest" support of the measure μ in the sense of the Hausdorff–Billingsley dimension w.r.t. ν^*. Let C be an arbitrary support of the measure μ. Then the set $C_1 = C \cap A$ is also a support of the same measure μ, and $C_1 \subset C$.

From $C_1 \subset A$, it follows that $\dim_H(C_1, \nu^*) \le D_0$, and

$$C_1 \subset A \subset A_3 = \left\{x : \liminf_{n\to\infty} \frac{\ln \mu(\Delta_n(x))}{\ln \nu^*(\Delta_n(x))} \ge D_0\right\}.$$

Therefore,

$$\dim_H(C_1, \nu^*) = \dim_H(C_1, \nu^*, \Phi_M) \ge D_0 \cdot \dim_H(C_1, \mu, \Phi_M) = D_0 \cdot 1 = D_0.$$

So, $\dim_H(C, \nu^*) \ge \dim_H(C_1, \nu^*) = D_0 = \dim_H(A, \nu^*)$. □

Corollary 4. *From the construction of the measure ν^* it follows that $\dim_H(S_\mu, \nu^*)$ is always equal to 1.*

Corollary 5. *The distribution function $h_1(t) := F_{\nu^*}(t) = \nu^*((-\infty, t)) = \nu^*([0, t))$ is the dimensional (gauge) function for the spectrum S_μ, and the probability measure ν^* coincides with the restriction of the measure H^{h_1} on the set S_μ.*

Corollary 6. *If the measure μ is absolutely continuous w.r.t. the measure ν^*, then $\dim_H(\mu, \nu^*) = 1$.*

Remark. From the latter theorem it follows that the Hausdorff-Billingsley dimension of the measure μ w.r.t. the measure ν^* can take any value from the set $[0, 1]$.

Examples.

If $p_{0k} = \frac{1}{2k}, \forall k \in N$, then $\dim_H(\mu, \nu^*) = 0$;
if $p_{0k} = \frac{1}{2} - \frac{1}{4\sqrt{k}}, \forall k \in N$, then $\dim_H(\mu, \nu^*) = 1$; but $\mu \perp \nu^*$.
if $p_{0,2k} = \frac{1}{2} - \frac{1}{4k}, p_{0,2k-1} = \frac{1}{2k}, \forall k \in N$, then $\dim_H(\mu, \nu^*) = \frac{1}{2}$.

Remark. To clarify how uniformly the measure μ is distributed on its spectrum, we shall firstly check whether $\mu \ll \nu^*$. To clarify how irregularly the measure μ is distributed on its spectrum, we should calculate the value of $\dim_H(\mu, \nu^*)$.

The measure with smaller dimension can be considered as more irregularly distributed on its spectrum.

To stress the difference between the spectrum S_μ and the set C_r of all incomplete sums of the second Ostrogradsky series, it is natural to use the Hausdorff-Billingsley dimension w.r.t. the measure ν_r which is uniformly distributed on the whole set C_r, i.e., ν_r is the probability distribution of the random variable

$$\xi_r = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\varepsilon_k}{q_k},$$

where ε_k are independent random variables taking the values 0 and 1 with probabilities $\frac{1}{2}$ and $\frac{1}{2}$ correspondingly.

Theorem 8. *The Hausdorff–Billingsley dimension of the measure μ_ξ w.r.t. ν_r is equal to*

$$\dim_H(\mu, \nu_r) = \liminf_{n\to\infty} \frac{H_n}{n \ln 2}. \tag{25}$$

The Hausdorff–Billingsley dimension of the spectrum S_μ w.r.t. ν_r is equal to

$$\dim_H(S_\mu, \nu_r) = \liminf_{k\to\infty} \frac{N_k}{k}, \tag{26}$$

where $N_k = \#\{j : j \leq k, p_{0j}p_{1j} > 0\}$.

Proof. The proof of the first assertion of the theorem is completely similar to the the proof of theorem 7. To prove the second statement, let us consider an auxiliary probability measure ν^* which was defined above.

Generally speaking, the measures μ and ν^* do not coincide (moreover, they can be mutually singular), but their spectra are the same. The measure ν^* is uniformly distributed on the spectrum of the initial measure. Therefore,

$$\dim_H(\nu^*, \nu_r) = \dim_H(S_{\nu^*}, \nu_r) = \dim_H(S_\mu, \nu_r).$$

So, to determine the Hausdorff–Billingsley dimension of the spectra of the initial measure w.r.t. the measure ν_r it is enough to apply the first statement of theorem 8 to the measure ν^*. □

Corollary 7. *The distribution function $h_2(t) := F_{\nu_r}(t) = \nu_r((-\infty, t)) = \nu_r([0, t))$ is the dimensional (gauge) function for the set C_r of all incomplete sums.*

Example. Let $p_{0k} = \frac{1}{2}$ if $k = 2^m$, and $p_{0k} = 0$ if $k \neq 2^m$, $m \in N$. In such a case:

$$\dim_H(\mu, \nu^*) = \dim_H(S_\mu, \nu^*) = 1,$$

and

$$\dim_H(\mu, \nu_r) = \dim_H(S_\mu, \nu_r) = 0.$$

7 Fractal dimension preservation

In this section we develop third approach how to study level of "irregularity" of probability distributions whose spectra are of zero Hausdorff dimension.

Let F be a probability distribution function and let $\gamma = \gamma_F$ be the corresponding probability measure with spectrum S_F. Let γ^* be the probability measure which are uniformly distributed on S_F.

We say that a distribution function F preserves the Hausdorff–Billingsley dimension (w.r.t. γ^*) on a set A if the Hausdorff–Billingsley dimension $\dim_H(E, \gamma)$ of any subset $E \subseteq A$ is equal to the Hausdorff–Billingsley dimension $\dim_H(E, \gamma^*)$.

If the probability distribution function F strictly increases (i.e., the spectrum S_F is a closed interval), then the above definition reduces to the usual definition of a transformation preserving the Hausdorff–Besicovitch dimension (see, e.g., [4] for details).

Let ξ and ξ^* be random variables defined by equalities (11) and (20) respectively, and let μ and ν^* be the corresponding probability measures. It is easily seen that their spectra coincide. If, in addition, $p_{ik} > 0$ for any $i \in \{0, 1\}$ and $k \in \mathbb{N}$, then $S_{\nu^*} = S_\mu = C_r$ with $r = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{q_k}$.

Remark. The distribution function F_μ does not preserve the classical Hausdorff–Besicovitch dimension, because $\dim_H(S_\mu) = 0$ and $\dim_H(F_\mu(S_\mu)) = 1$. But, if the random variable ξ is distributed "regularly" on its spectrum, then F_μ can preserve the Hausdorff–Billingsley dimension on S_μ.

Theorem 9. *If there exists a positive constant p_0 such that $p_{ik} > p_0$, then the distribution function F_ξ of the random variable ξ preserves the Hausdorff–Billingsley dimension $\dim_H(\cdot, \mu)$ on the spectrum S_μ if and only if the Hausdorff–Billingsley dimension of the measure μ with respect to the measure ν^* is equal to 1, i.e., if*

$$\liminf_{n\to\infty} \frac{H_n}{g_n \ln 2} = 1. \tag{27}$$

Proof. Necessity. Let $\dim_H(\mu, \nu^*) = \inf\{\dim_H(E, \nu^*),\quad E \in \mathcal{B}, \mu(E) = 1\}$ be the Hausdorff–Billingsley dimension of the measure μ w.r.t. the measure ν^*. From Theorem 7 it follows that $\dim_H(\mu, \nu^*) = \liminf\limits_{n\to\infty} \frac{H_n}{g_n \ln 2}$ with $H_n = h_1 + h_2 + \cdots + h_n$ and $h_k = -(p_{0k} \ln p_{0k} + p_{1k} \ln p_{1k})$. If $\dim_H(\mu, \nu^*) < 1$, then there exists a support E of the measure μ such that $\dim_H(\mu, \nu^*) \le \dim_H(E, \nu^*) < 1$. Since $\mu(E) = 1$, we conclude that $\dim_H(E, \mu) = 1 \neq \dim_H(E, \nu^*)$, which contradicts the assumption of the theorem.

Sufficiency. It is easy to check n that $h_k \le \ln 2$ for any $k \in \mathbb{N}$ and the equality holds if and only if $p_{0k} = p_{1k} = \frac{1}{2}$. Since $p_{ik} > 0, \forall i \in \{0, 1\}, \forall k \in N$, we have $g_n = n, \forall n \in N$. Therefore, condition (27) is equivalent to the existence of the following limit

$$\lim_{k\to\infty} \frac{h_1 + h_2 + \cdots + h_k}{k \ln 2} = 1. \tag{28}$$

For a given $\varepsilon \in (0, \frac{1}{10})$ let us consider the sets

$$M^+_{\varepsilon,k} = \left\{ j : j \in \mathbb{N},\, j \leq k,\, \left| p_{0j} - \frac{1}{2} \right| \leq \varepsilon \right\},$$

and

$$M^-_{\varepsilon,k} = \{1, 2, \ldots, k\} \setminus M^+_{\varepsilon,k} = \left\{ j : j \in \mathbb{N},\, j \leq k,\, \left| p_{0j} - \frac{1}{2} \right| > \varepsilon \right\}.$$

From $h_k \leq \ln 2$ and condition (28), it follows that

$$\lim_{k\to\infty} \frac{|M^+_{\varepsilon,k}|}{k} = 1, \qquad \lim_{k\to\infty} \frac{|M^-_{\varepsilon,k}|}{k} = 0.$$

Let $\Delta_{\alpha_1(x)\ldots\alpha_k(x)}$ be the k-rank cylindrical interval containing the point $x \in S_\mu$, and let us consider the limit

$$V(x) = \lim_{k\to\infty} \frac{\ln \mu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \nu^*(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})},\ x \in S_\mu$$

Firstly, we show that from $p_{ij} > p_0$ and from condition (27), it follows that the above limit exists and $V(x) = 1$ for any $x \in S_\mu$.

$$\frac{\ln \mu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \nu^*(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = \frac{\ln(p_{\alpha_1(x)1} \cdot \cdots \cdot p_{\alpha_k(x)k})}{\ln 2^{-k}} = \frac{\sum\limits_{j\in M^+_{\varepsilon,k}} \ln p_{\alpha_j(x)j} + \sum\limits_{j\in M^-_{\varepsilon,k}} \ln p_{\alpha_j(x)j}}{-k \ln 2}.$$

If $j \in M^+_{\varepsilon,k}$, then $\frac{1}{2} - \varepsilon \leq p_{\alpha_j(x)j} \leq \frac{1}{2} + \varepsilon$, and, consequently, there exists a number $C_{\varepsilon,k} \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ such that

$$\sum_{j\in M^+_{\varepsilon,k}} \ln p_{\alpha_j(x)j} = |M^+_{\varepsilon,k}| \cdot \ln C_{\varepsilon,k}.$$

If $j \in M^-_{\varepsilon,k}$, then $p_0 \leq p_{\alpha_j(x)j} \leq 1 - p_0$. Therefore, there exists a number $d_{\varepsilon,k} \in [p_0, 1 - p_0]$ such that

$$\sum_{j\in M^-_{\varepsilon,k}} \ln p_{\alpha_j(x)j} = |M^-_{\varepsilon,k}| \cdot \ln d_{\varepsilon,k}.$$

Let

$$V^*(x) := \limsup_{k\to\infty} \frac{\ln \mu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \nu^*(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} = \limsup_{k\to\infty} \left(\frac{|M^+_{\varepsilon,k}|}{k} \cdot \frac{\ln C_{\varepsilon,k}}{-\ln 2} + \frac{|M^-_{\varepsilon,k}|}{k} \cdot \frac{\ln d_{\varepsilon,k}}{-\ln 2} \right).$$

We have $V^*(x) \leq \frac{\ln\left(\frac{1}{2}-\varepsilon\right)}{-\ln 2}$, because $\frac{|M^+_{\varepsilon,k}|}{k} \to 1$, $\frac{1}{2} - \varepsilon \leq C_{\varepsilon,k} \leq \frac{1}{2} + \varepsilon$, $\frac{|M^-_{\varepsilon,k}|}{k} \to 0$, $p_0 \leq d_{\varepsilon,k} \leq 1 - p_0$.

In the same way we can show that

$$V_*(x) = \liminf_{k\to\infty} \frac{\ln \mu(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \nu^*(\Delta_{\alpha_1(x)\ldots\alpha_k(x)})} \geq \frac{\ln\left(\frac{1}{2}+\varepsilon\right)}{-\ln 2}, \quad \forall x \in S_\nu, \quad \forall \varepsilon \in \left(0, \frac{1}{10}\right).$$

Therefore $V(x) = 1$ for any $x \in S_\mu$, and, by using Billingsley's theorem [8], we have for all $E \subset S_\mu$:

$$\dim_H(E, \nu^*, \Phi_M) = 1 \cdot \dim_H(E, \mu, \Phi_M),$$

where Φ_M is a fine covering family of cylindrical intervals $\Delta_{\alpha_1(x)\ldots\alpha_k(x)}$ for the spectrum S_μ. The image of the family Φ_M under the distribution function F_{ν^*} coincides with the family of all binary subintervals of the unit interval. So, from the fractality of the family $F_{\nu^*}(\Phi_M)$ and from lemma 4 it follows that $\dim_H(E, \nu^*, \Phi_M) = \dim_H(E, \nu^*)$. The image of the family Φ_M under the distribution function F_μ coincides with the family of all Q^*-cylinders of the Q^*-expansion of real numbers with $q_{ik} = p_{ik}$ (see, e.g., [7] for details). Since $p_{ik} > p_0 > 0$, from lemma 1 of the paper [7] and from our lemma 4 it follows that $\dim_H(E, \mu, \Phi_M) = \dim_H(E, \mu)$. So, $\dim_H(E, \nu^*) = \dim_H(E, \mu)$, $\forall\, E \subset S_\mu$, which proves the theorem. □

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